

ON CONTINUOUS CAUSAL ISOMORPHISMS

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ABSTRACT. It is shown that continuous causal isomorphisms on two-dimensional Minkowski spacetime can be characterized by the invariance of wave equations.

1. INTRODUCTION

In 1964, Zeeman has shown that general form of causal isomorphism F defined on \mathbb{R}_1^n , when $n \geq 3$, has the form $F(\mathbf{x}) = aA\mathbf{x} + \mathbf{b}$, where a is a positive real number and A is an orthochronous matrix. ([1]) As Zeeman remarked, his result does not hold when $n = 2$. In 2010, the general form of causal isomorphism defined on \mathbb{R}_1^2 was clarified in [2] and [3]. When these results were obtained, the results strongly suggested that there are some kind of close relationship between causal isomorphism and wave equation and thus, in [4], Low officially proposed the question how to characterize causal isomorphisms on \mathbb{R}_1^2 in terms of wave equations. In [5], it is shown that, when $n \geq 3$, causal isomorphisms on \mathbb{R}_1^n can be characterized by invariance of wave equations and in [6], even if $n = 2$, causal isomorphisms can be characterized by the invariance of wave equations. However, one of characteristic differences between causal isomorphisms on \mathbb{R}_1^n ($n \geq 3$) and \mathbb{R}_1^2 is that causal isomorphisms on \mathbb{R}_1^2 are not necessarily differentiable whereas they are C^∞ when $n \geq 3$. Since causal isomorphisms are necessarily smooth when $n \geq 3$, in [5], we don't need to worry about the smoothness of causal isomorphisms. In contrast, in [6], it must be explicitly assumed that the causal isomorphisms are at least C^2 to ensure that we can apply wave operators since there are non-differentiable causal isomorphisms. In other words, it only remains that how we can characterize C^0 and C^1 causal isomorphisms on \mathbb{R}_1^2 in terms of wave equations.

In this paper, it is shown that when $n = 2$, even continuous causal isomorphisms can be characterized by the invariance of wave equations. In order to take derivatives of C^0 functions, we need to generalize the derivatives and to this end, it is necessary to introduce generalized functions and generalized derivatives.

Key words and phrases. causal isomorphism, causal relation, wave equation, Zeeman theorem.

2. AN OVERVIEW ON DISTRIBUTIONS

To take derivatives of functions which are not differentiable in the classical sense, we need to generalize the notion of functions and derivatives. In this section, we introduce distributions (or, generalized functions) and their derivatives, and briefly review their basics.

Ordinarily, a real-valued function f is given by specifying its value $f(x)$ to each point x in the domain of definition. However, f can be defined in another equivalent ways. For example, let f be a continuous function on \mathbb{R}^2 and assume that we know the value of $\int_{\mathbb{R}^2} f(x, y)\phi(x, y)dxdy$ for each C^∞ function ϕ whose support is compact. Then, by use of convolution, we can obtain the value $f(x, y)$ for each $(x, y) \in \mathbb{R}^2$. (For details, see Theorem 7.7 in [7]). Therefore, the value of $\int_{\mathbb{R}^2} f(x, y)\phi(x, y)dxdy$ for each C^∞ function ϕ whose support is compact, can be used to define a function and this gives us a way to generalize the concept of ordinary functions. Hence, we proceed as follows.

Definition. By test functions on \mathbb{R}^2 , we mean C^∞ functions defined on \mathbb{R}^2 with compact supports and we denote the set of all test functions by \mathcal{D} . A distribution or a generalized function on \mathbb{R}^2 is a mapping $F : \mathcal{D} \rightarrow \mathbb{R}$ such that F is \mathbb{R} -linear and satisfies the following continuity condition. : Let $\phi_k \in \mathcal{D}$ and $\text{supp } \phi_k$ be contained in a fixed bounded set for all k . If ϕ_k and all their derivatives converge uniformly to zero, then $F(\phi_k)$ converges to zero.

Motivated by the above, from now on, we identify locally integrable function f defined on \mathbb{R}^2 with the distribution $\phi \mapsto \int f\phi d\mu$ where $d\mu = dxdy$ is the Lebesgue measure. Since we are interested only in continuous functions in this paper, and continuous functions are locally integrable, we identify continuous function f with the distribution $\phi \mapsto \int f\phi d\mu$ and vice versa. Therefore, when we say that $f = g$ as distributions, it means that $\int f\phi d\mu = \int g\phi d\mu$ for all $\phi \in \mathcal{D}$.

We now generalize the notion of derivatives of ordinary functions to get derivatives of distributions. If f is C^1 , integration by parts gives us $\int_{\mathbb{R}^2} \frac{\partial f}{\partial x} \phi dxdy = \int_{\mathbb{R}} \left(\left[f(x, y)\phi(x, y) \right]_{x=-\infty}^{x=\infty} - \int_{\mathbb{R}} f \frac{\partial \phi}{\partial x} dx \right) dy$. Since test function ϕ has a compact support, we have $\phi(-\infty, y) = \phi(\infty, y) = 0$ and thus $\int_{\mathbb{R}^2} \frac{\partial f}{\partial x} \phi dxdy = - \int_{\mathbb{R}^2} f \frac{\partial \phi}{\partial x} dxdy$. Motivated by this fact, we define the derivatives of distributions as the following.

Definition. If F is a distribution on \mathbb{R}^2 , we define its partial derivatives by $\frac{\partial F}{\partial x}(\phi) = -F(\frac{\partial \phi}{\partial x})$ and $\frac{\partial F}{\partial y}(\phi) = -F(\frac{\partial \phi}{\partial y})$.

From the definition of distribution, it is not difficult to see that the derivatives of a distribution are also distributions, and it must be noted that any distribution can be differentiated infinitely many times since test

functions are differentiable infinitely many times. For example, if f is a continuous function defined on \mathbb{R}^2 , then its partial derivative $\frac{\partial f}{\partial x}$ is the distribution $\phi \mapsto -\int_{\mathbb{R}^2} f(x, y) \frac{\partial \phi}{\partial x} dx dy$. We also remark that for any distribution F , we have $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$. To compare the derivative of a distribution with the derivative in the classical sense, we have the following.

Theorem 2.1. *Let f be a real-valued function defined on \mathbb{R} , which is differentiable in the classical sense. Then, its derivative f' in the classical sense is the same as the derivative in the distribution sense.*

Proof. Let \hat{f} denote the derivative of f in the distribution sense. To show that $f' = \hat{f}$, it is sufficient to show that they produce the same value when multiplied by test functions and integrated. By computation, we have $\int_{\mathbb{R}} f' \phi dx = -\int_{\mathbb{R}} f \phi' dx$ which is the same as $\hat{f}(\phi) = -\int_{\mathbb{R}} f \phi' dx$ by definition. \square

We finally remark that by the same argument, we can show that the above theorem also holds for multi-variable functions and for their partial derivatives. For example, though $f(x, y) = \sqrt{x^2 + y^2}$ does not have partial derivatives at $(0, 0)$ in the classical sense, we can see that its distribution $\frac{\partial f}{\partial x}$ is a map $\phi \mapsto -\int \sqrt{x^2 + y^2} \frac{\partial \phi}{\partial x} dx dy$ and it is not difficult to see that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ as distributions. This is the way in which we generalize derivatives.

3. CONTINUOUS CAUSAL ISOMORPHISMS

From now on, all functions are considered as identified with the corresponding distributions and derivatives are considered in the distribution sense.

Proposition 3.1. *Let $z = f(x, y)$ where f is locally integrable. Then we have $\frac{\partial f}{\partial x} = 0$ if and only if $z = f(y)$.*

Proof. Assume that $\frac{\partial f}{\partial x} = 0$ and choose a C^∞ function $\phi_0 : \mathbb{R} \rightarrow \mathbb{R}$ such that $\text{supp } \phi_0$ is compact and $\int_{-\infty}^{\infty} \phi_0 dx = 1$. For $\phi \in \mathcal{D}$, let $\psi(x, y) = \phi(x, y) - \phi_0(x) \int_{-\infty}^{\infty} \phi(u, y) du$. Then, it is easy to see that $\psi \in \mathcal{D}$. If we let $\eta_\phi(x, y) = \int_{-\infty}^x \phi(u, y) du - \left(\int_{-\infty}^{\infty} \phi(u, y) du \right) \int_{-\infty}^x \phi_0(u) du$, then we have $\frac{\partial \eta_\phi}{\partial x}(x, y) = \psi(x, y)$ and $\eta_\phi \in \mathcal{D}$ since $\int_{-\infty}^{\infty} \phi_0 dx = 1$.

Since $\frac{\partial f}{\partial x} = 0$, we have $0 = \int \frac{\partial f}{\partial x} \eta_\phi d\mu = -\int f \frac{\partial \eta_\phi}{\partial x} d\mu = -\int f \psi d\mu$. If we substitute $\psi(x, y) = \phi(x, y) - \phi_0(x) \int_{-\infty}^{\infty} \phi(u, y) du$ into $\int f \psi d\mu = 0$, we

have

$$\begin{aligned}
\int_{\mathbb{R}^2} f \phi \, d\mu &= \int_{\mathbb{R}^2} f(x, y) \phi_0(x) \left(\int_{-\infty}^{\infty} \phi(u, y) du \right) d\mu \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \phi_0(x) \phi(u, y) \, du dx dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u, y) \phi_0(u) \phi(x, y) \, du dx dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(u, y) \phi_0(u) du \right] \phi(x, y) \, dx dy.
\end{aligned}$$

Therefore, we have $f = \int_{-\infty}^{\infty} f(u, y) \phi_0(u) \, du$, which is a function of y .

Conversely, we now assume that $z = f(y)$. Then, we have

$$\begin{aligned}
\frac{\partial f}{\partial x} &= -f \left[\frac{\partial \phi}{\partial x} \right] \\
&= - \int \int f \frac{\partial \phi}{\partial x} \, dx dy \\
&= - \int_{-\infty}^{\infty} f(y) \left(\int_{-\infty}^{\infty} \frac{\partial \phi}{\partial x} \, dx \right) dy \\
&= - \int_{-\infty}^{\infty} f(y) \cdot 0 \, dy = 0.
\end{aligned}$$

In the last equality, we have used the fact that ϕ has a compact support. \square

Theorem 3.1. *Let f and g be locally integrable functions defined on \mathbb{R}^2 . If $\frac{\partial f}{\partial x} = g$, then there exist distributions h and G such that $f = h + G$, where $\frac{\partial h}{\partial x} = 0$ and $\frac{\partial G}{\partial x} = g$.*

Proof. Choose a smooth $\phi_0 : \mathbb{R} \rightarrow \mathbb{R}$ such that $\int_{-\infty}^{\infty} \phi_0 \, dx = 1$ and $\text{supp } \phi_0$ is compact. If we define ψ and η_ϕ in the same manner as in the Proposition 3.1, then, we have $\psi \in \mathcal{D}$, $\eta_\phi \in \mathcal{D}$ and $\frac{\partial \eta_\phi}{\partial x}(x, y) = \psi(x, y)$.

Since $\frac{\partial f}{\partial x} = g$, we have

$$\int f \frac{\partial \phi}{\partial x} \, d\mu = - \int g \phi \, d\mu, \quad \text{for all } \phi \in \mathcal{D}.$$

In particular, we have

$$\begin{aligned}
\int f \frac{\partial \eta_\phi}{\partial x} \, d\mu &= - \int g \eta_\phi \, d\mu \\
\text{and so } \int f \psi \, d\mu &= - \int g \eta_\phi \, d\mu.
\end{aligned}$$

If we substitute $\psi(x, y) = \phi(x, y) - \phi_0(x) \int_{-\infty}^{\infty} \phi(u, y) du$ into the above equation, we have

$$\int_{\mathbb{R}^2} f \phi \, d\mu - \int_{\mathbb{R}^2} f(x, y) \phi_0(x) \left(\int_{-\infty}^{\infty} \phi(u, y) \, du \right) d\mu = - \int g \eta_{\phi} \, d\mu \cdots (*).$$

If let $h(\phi) = \int_{\mathbb{R}^2} \left(\int_{-\infty}^{\infty} f(u, y) \phi_0(u) \, du \right) \phi(x, y) \, d\mu$, which is the same as the second term on the left hand side of (*), then it is easy to see that h is a distribution and, since $h = \int_{-\infty}^{\infty} f(u, y) \phi_0(u) \, du$ as a function, we have $\frac{\partial h}{\partial x} = 0$ by Proposition 3.1.

If we consider $G : \phi \mapsto - \int g \eta_{\phi} \, d\mu$, then, from (*), $G = f - h$ and so G is a distribution.

To show $\frac{\partial G}{\partial x} = g$, since $\int_{-\infty}^{\infty} \frac{\partial \phi}{\partial u}(u, y) \, du = 0$, we have

$$\begin{aligned} \frac{\partial G}{\partial x}(\phi) &= -G\left(\frac{\partial \phi}{\partial x}\right) \\ &= \int g \left[\int_{-\infty}^x \frac{\partial \phi}{\partial u}(u, y) \, du - \left(\int_{-\infty}^{\infty} \frac{\partial \phi}{\partial u}(u, y) \, du \right) \int_{-\infty}^x \phi_0(u) \, du \right] d\mu \\ &= \int g(x, y) \phi(x, y) \, d\mu, \text{ since } \phi \text{ has compact support.} \end{aligned}$$

Therefore, we have $\frac{\partial G}{\partial x} = g$. □

We now prove one of the key theorems which characterize continuous causal isomorphisms.

Theorem 3.2. *Let $f = f(x, y)$ be a locally integrable function. Then, $\frac{\partial^2 f}{\partial x \partial y} = 0$ if and only if $f(x, y) = \alpha(x) + \beta(y)$, where α and β are locally integrable functions. Furthermore, if f is continuous, then α and β are continuous.*

Proof. Assume that $\frac{\partial^2 f}{\partial x \partial y} = 0$ and choose a C^∞ function $\phi_0(x)$ such that $\int_{-\infty}^{\infty} \phi_0 \, dx = 1$ and the support of ϕ_0 is compact. Let $\phi_1(x, y) = -\phi_0(x)\phi_0(y)$. Then, ϕ_1 is C^∞ , has compact support, and we have $\int_{-\infty}^{\infty} \phi_1(u, y) \, du = -\phi_0(y)$ and $\int_{-\infty}^{\infty} \phi_1(x, u) \, du = -\phi_0(x)$. Given any test function ϕ , define ψ and η_{ϕ} by

$$\psi(x, y) = \phi(x, y) - \phi_0(x) \int_{-\infty}^{\infty} \phi(u, y) \, du - \phi_0(y) \int_{-\infty}^{\infty} \phi(x, v) \, dv - \phi_1(x, y) \left(\int \phi \, d\mu \right)$$

and

$$\begin{aligned}\eta_\phi(x, y) &= \int_{-\infty}^x \int_{-\infty}^y \phi(u, v) dv du - \int_{-\infty}^x \phi_0(u) du \int_{-\infty}^\infty \int_{-\infty}^y \phi(u, v) dv du \\ &\quad - \int_{-\infty}^y \phi_0(u) du \int_{-\infty}^\infty \int_{-\infty}^x \phi(u, v) dv du - \int_{-\infty}^x \int_{-\infty}^y \phi_1(u, v) dv du \left(\int \phi d\mu \right)\end{aligned}$$

In the definition of η_ϕ , we can see that, for large x , the first and second terms cancel out, and the third and fourth terms cancel out. Also, for large y , the first and third terms cancel out and the second and fourth terms cancel out. Therefore, η_ϕ has a compact support. Now it is easy to see that ψ and η_ϕ are test functions and $\frac{\partial^2 \eta_\phi}{\partial x \partial y} = \psi(x, y)$. Since $f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = 0$, we have $0 = \int \int f_{xy} \phi(x, y) dx dy = \int \int f(x, y) \phi_{xy} dx dy$ for any test functions $\phi(x, y)$. Therefore, we have $0 = \int \int f(x, y) \frac{\partial^2 \eta_\phi}{\partial x \partial y} dx dy = \int \int f(x, y) \psi(x, y) dx dy$.

If we substitute the definition of ψ into $\int f \psi d\mu = 0$, we have

$$\begin{aligned}\int \int f \phi dx dy &= \int \int f(x, y) \phi_0(x) \left(\int_{-\infty}^\infty \phi(u, y) du \right) dx dy \\ &\quad + \int \int f(x, y) \phi_0(y) \left(\int_{-\infty}^\infty \phi(x, v) dv \right) dx dy \\ &\quad + \int \int f(x, y) \phi_1(x, y) dx dy \left(\int \phi d\mu \right) \\ &= \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty f(x, y) \phi_0(x) \phi(u, y) du dx dy \\ &\quad + \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty f(x, y) \phi_0(y) \phi(x, v) dv dx dy \\ &\quad + \left(\int_{-\infty}^\infty \int_{-\infty}^\infty f(x, y) \phi_1(x, y) dx dy \right) \left(\int_{-\infty}^\infty \int_{-\infty}^\infty \phi dx dy \right) \\ &= \int_{-\infty}^\infty \int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(u, y) \phi_0(u) du \right] \phi(x, y) dx dy \\ &\quad + \int_{-\infty}^\infty \int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(x, v) \phi_0(v) dv \right] \phi(x, y) dx dy \\ &\quad + \int_{-\infty}^\infty \int_{-\infty}^\infty c \phi(x, y) dx dy, \\ \text{where } c &= \int_{-\infty}^\infty \int_{-\infty}^\infty f(x, y) \phi_1(x, y) dx dy.\end{aligned}$$

In other words, $f(x, y) = \alpha(x) + \beta(y)$ where $\alpha(x) = \int_{-\infty}^\infty f(x, v) \phi_0(v) dv$ and $\beta(y) = \int_{-\infty}^\infty f(u, y) \phi_0(u) du + c$. Since f is locally integrable, from the

definition of α and β we can see that α and β are locally integrable. Since any continuous map defined on compact set is uniformly continuous, if f is continuous, then α and β are continuous.

Since $f_{xy} = f_{yx}$ for any distribution f , the converse is easily obtained from Proposition 3.1. \square

4. THE CHARACTERIZATION

On two-dimensional Minkowski spacetime \mathbb{R}_1^2 , if we use null coordinates $u = x + t$ and $v = x - t$, then we have the following.

Theorem 4.1. *Let $(U, V) = F(u, v)$ be a causal isomorphism on \mathbb{R}_1^2 . Then, there exist unique homeomorphisms φ and ψ on \mathbb{R} , which are both increasing or both decreasing such that if φ and ψ are increasing, then we have $F(u, v) = (\varphi(u), \psi(v))$, or if φ and ψ are decreasing, then we have $F(u, v) = (\varphi(v), \psi(u))$.*

Conversely, for any given homeomorphisms φ and ψ on \mathbb{R} , which are both increasing or both decreasing, the function F defined as above is a causal isomorphism on \mathbb{R}_1^2 .

Proof. See Theorem 2.2 in [6]. \square

We now begin to characterize continuous causal isomorphisms by use of results obtained in the previous section.

Theorem 4.2. *Let $(\sigma, \tau) = F(u, v)$ be a homeomorphism from \mathbb{R}_1^2 onto \mathbb{R}_1^2 where (u, v) and (σ, τ) are null coordinates. Suppose that, for any function θ on \mathbb{R}_1^2 , $\theta_{uv} = 0$ if and only if $\theta_{\sigma\tau} = 0$. Then, there exist homeomorphisms φ and ψ on \mathbb{R} such that, either $F(u, v) = (\varphi(u), \psi(v))$ or $F(u, v) = (\varphi(v), \psi(u))$.*

Proof. If we let $\theta(\sigma, \tau) = \sigma$, then θ satisfies $\theta_{\sigma\tau} = 0$ and so, by assumption, we have $\theta_{uv} = 0$. Then, by Theorem 3.2, there exist continuous functions α and β such that $\sigma(u, v) = \alpha(u) + \beta(v)$. Likewise, by considering $\theta(\sigma, \tau) = \tau$, we can get continuous functions γ and δ such that $\tau(u, v) = \gamma(u) + \delta(v)$. From Proposition 3.1, we can see that $\theta(\sigma, \tau) = \sigma^2$ satisfies $\theta_{\sigma\tau} = 0$ and thus, we have $\frac{\partial \sigma^2}{\partial u \partial v} = 0$. Since $\sigma(u, v)^2 = \alpha(u)^2 + 2\alpha(u)\beta(v) + \beta(v)^2$, and $\frac{\partial^2}{\partial u \partial v}(\alpha(u)^2 + \beta(v)^2) = 0$ by Theorem 3.2, we have $\frac{\partial^2}{\partial u \partial v}(\alpha(u)\beta(v)) = 0$. By Theorem 3.2 again, we have $\alpha(u)\beta(v) = f(u) + g(v)$, where f and g are continuous. If we assume that α is not constant, there exist u_1 and u_2 such that $\alpha(u_1) \neq \alpha(u_2)$ and so we have

$$\begin{aligned} \alpha(u_1)\beta(v) &= f(u_1) + g(v) \text{ and} \\ \alpha(u_2)\beta(v) &= f(u_2) + g(v). \end{aligned}$$

Therefore, we have $\beta(v) = \frac{f(u_1) - f(u_2)}{\alpha(u_1) - \alpha(u_2)}$. In other words, if α is not constant, then β is a constant function. Since $(\sigma, \tau) = F(u, v)$ is a bijection, if α is a constant function, then β is not a constant function.

By considering $\theta(\sigma, \tau) = \tau^2$, by the same argument as above, we can show that γ is not a constant function if and only if δ is a constant function.

Assume first that α is not a constant function. Then, by the previous observation, we have $\sigma = \varphi(u)$ where $\varphi(u) = \alpha(u) + \text{constant}$ is a continuous function. If $\gamma(u)$ is not a constant function, then $\tau = \gamma(u) + c$ and thus F is not a bijection, which is a contradiction. Therefore, γ is a constant function and thus we have $\tau = \psi(v)$ where $\psi(v) = \delta(v) + \text{constant}$ is a continuous function. Since F is bijective, φ and ψ must be bijective and by the topological domain of invariance, continuous bijections φ and ψ are homeomorphisms on \mathbb{R} .

We now assume that α is a constant function. Then, since β is not a constant function, we have $\sigma = \varphi(v)$ where $\varphi(v) = \beta(v) + \text{constant}$ is a continuous function and, by the same argument as the above, we can show that $\tau = \psi(u)$ where $\psi(u) = \gamma(u) + \text{constant}$ is a continuous function. By the same argument as the above, φ and ψ are homeomorphisms on \mathbb{R} . \square

By combining the above two theorems, we have the following.

Theorem 4.3. *Let $F : \mathbb{R}_1^2 \rightarrow \mathbb{R}_1^2$ be a homeomorphism given by $(\sigma, \tau) = F(u, v)$ where (u, v) and (σ, τ) are null coordinates. Then, a necessary and sufficient condition for F to be a causal isomorphism is that, for any continuous function θ on \mathbb{R}_1^2 , we have $\theta_{uv} = 0$ if and only if $\theta_{\sigma\tau} = 0$, and either (1) σ and τ are increasing functions of u and v , respectively, or (2) σ and τ are decreasing functions of v and u , respectively.*

Since \mathbb{R}_1^2 is strongly causal, any causal isomorphism on \mathbb{R}_1^2 is a homeomorphism and thus, this theorem characterizes all of the causal isomorphisms on \mathbb{R}_1^2 by invariance of wave equations, regardless of their smoothness. With this theorem and the results in [5] and [6], the characterization of causal isomorphisms in terms of wave equations is completed regardless of their smoothness and their dimensions.

5. ACKNOWLEDGEMENT

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